Eur. J. Mech. B/Fluids, **18**, (1), 93-102, 1999

Solitary wave decay in non-linear Faraday resonance

I. BAKHOLDIN *[a], A. IL'ICHEV *[b]

ABSTRACT. – This investigation studies possible mechanisms by which localised disturbances in a vertically oscillating long channel undergo transition to instability. Modulated cross-waves with a carrier frequency close to natural are considered. The channel is subjected to oscillations with the frequency shifted slightly below twice the natural value. The waves are considered to be described by the Miles equation (Miles, 1984). We report here numerical investigations of the instability of various solitary wave disturbances for a range of controlling parameters. Certain mechanisms describing such transitions have been predicted theoretically by Il'ichev (1998). © Elsevier, Paris.

Keywords. – Faraday resonance, solitary waves, transition to instability.

1. Introduction

Wu *et al.* (1984) reported the experimental observation of a solitary standing surface wave in a forced rectangular tank. This observation prompted a great number of studies of Faraday resonance, of interest because of the fundamental importance of the solitary wave in non-linear wave theory.

The theoretical model, describing surface waves in a long channel, either subjected to a disturbance with a frequency close to the natural value, or vertically oscillating with a frequency close to twice to natural frequency, was given by Miles, (1984) and Lazzara and Putterman, (1984) in the form of the equation for the envelope of non-linear modulated cross-waves. For a vertically oscillating channel (subjected to oscillations $\zeta = A\cos 2\omega t$) this equation has the form of the modified Schrodinger equation (Miles, 1984)

$$iu_t + \beta u + u_{xx} \pm 2|u|^2 u + i\alpha u + \gamma u^* = 0 \tag{1.1}$$

where suffices x and t denote differentiation with respect to the slow spatial and temporal variables, and an asterisk denotes the complex conjugate. Equation (1.1) is derived under the assumption that

$$k\eta^0 = \varepsilon \ll 1, \quad \frac{\omega_1^2}{\omega^2} - 1 = 2\varepsilon^2\beta, \quad \frac{A\omega^2}{g} = \varepsilon^2\gamma, \quad \widehat{\delta} = \varepsilon^2\alpha, \quad \alpha, \beta, \gamma = O(1)$$

where η^0 is a characteristic amplitude of a surface disturbance, $\omega_1 = 2gkT$, $T = \tanh kd$, $k = \pi/b$, b and d are respectively the width and depth of the channel, $\widehat{\delta}$ is the damping ratio, and g is the gravitational acceleration. For example, for a channel or a long tank with a depth and width of several centimeters, the forcing amplitude A should have a magnitude about one millimeter and a forcing frequency of about 10 Hz.

The slow dimensionless variables x and t are related to the physical ones x_{ph} and t_{ph} as follows

$$t = \varepsilon^2\omega t_{ph}, \quad x = \frac{\varepsilon k x_{ph}}{\sqrt{K}}, \quad K = \frac{1}{4T}(T + kd\,\mathrm{sech}^2\, kd).$$

* Permanent address: Mathematical Steklov Inst., Gubkina str. 8, 117966 Moscow GSP-1, Russia; Correspondence and reprints – Prof A. Il'ichev.
[a] Keldysh Institute of Applied Mathematics, Russian Ac. Sci. Miusscaya sq. 4, 125047 Moscow, Russia.
[b] Mathematisches institut A, Universität Stuttgart Pfaffenwaldring 57, 70569 Stuttgart, Germany.

The envelope $u = u(x,t)$ in (1.1) is related to the dominant lowest cross mode of the water waves within the tank via

$$u = \sqrt{2N}\eta_1, \quad N = \frac{k^2}{64T^4}(6T^6 - 5T^4 + 16T^2 - 9),$$

where η_1 is the envelope of (0,1) mode of the surface disturbance η:

$$\eta = \varepsilon\eta_1 \cos ky \exp(-i\omega t_{ph}) + c.c + O(\varepsilon^2).$$

We consider here only the modulationally unstable case (sign + at the cubic nonlinearity in (1.1)). The modulational instability takes place when $N > 0$, which implies $kd > 1.022$ (Miles, 1984).

The presence of the forcing term in (1.1) breaks the rotational symmetry of the problem and consequently the solitary wave solution of (1.1) is a standing (not travelling) wave (Miles, 1984):

$$u = r\exp(i\psi), \quad r = \rho \operatorname{sech} \rho x, \quad \rho = \sqrt{-\beta - \gamma\cos 2\psi}$$
$$\psi = \psi_{1,2}, \quad \psi_1 = \frac{\pi}{2} - \frac{1}{2}\arcsin(\alpha/\gamma), \quad \psi_2 = \frac{1}{2}\arcsin(\alpha/\gamma). \tag{1.2}$$

It was pointed out by Miles (1984) that the choice $\psi = \psi_2$ always leads to an unstable solitary wave. In Ladeke and Spatschek (1991) the stability of the solitary wave solution is considered for various ranges of parameters. It was concluded that the solitary wave (1.2) with $\psi = \psi_1$ is stable for $\beta < 0$ and $\alpha^2 < \gamma^2 < \alpha^2 + \beta^2$. The region of stability of the solitary wave (1.2), as understood by Laedke and Spatschek (1991) coincides with the region of stability for the quiescent state. The rather trivial analysis implies the following instability criterion for the quiescent state.

Case 1. $\beta < 0$.
The quiescent state is stable when either $\gamma^2 < \beta^2$, or $\gamma^2 > \beta^2$ and simultaneously $\beta^2 + \alpha^2 > \gamma^2$.

Case 2. $\beta > 0$.
The quiescent state is stable when $\alpha > \gamma$.

However, according to the results of Barashenkov *et al.*, (1991), there exists a region in the domain $\alpha^2 < \gamma^2 < \alpha^2 + \beta^2$, $\beta < 0$ with a border line at which a Hopf bifurcation takes place, and temporal oscillatory instability appears. The existence of periodic standing waves in the region $\gamma^2 > \alpha^2 + \beta^2$ for $\beta < 0$ indicates the specific feature of the transition to instability of a solitary wave in this region: the instability has a spatial character and the final stage of solitary wave instability is a cnoidal-wave type solution (Laedke and Spatschek, 1991). Note, that in this region the quiescent state is unstable, yet the periodic wave appears to be stable. In Il'ichev (1998) asymptotic formulae were obtained for various new solutions of (1.1) of the standing wave type, bifurcating from the critical curve $\gamma^2 = \alpha^2 + \beta^2$ in different parameter ranges. Among them there are new types of solitary wave and also the so called generalized solitary waves with non-decreasing ripples at infinity. In the dispersion systems with conservation of energy the existence of the generalized solitary waves implies a special behaviour of the evolution of localized disturbances. Namely, no localized initial disturbance can propagate steadily. This implies in turn the decay of localized disturbances due to permanent radiation of oscillatory waves (see e.g. Benilov *et al.*, 1993; Bakholding and Il'ichev, 1996). In Il'ichev (1998) it was conjectured that the system described by equation (1.1), which is in fact not a conservation one, will demonstrate similar features at the first stages of the transition to instability of the localized disturbances in the region where the generalized solitary waves exist. The numerical results, reported here, show that this conjecture is valid.

The transition to instability in oscillating tanks frequently leads to the following chaotic behaviour of localized disturbances. Not presuming to summarize all the literature devoted to chaotic phenomena taking place in the nonlinear Faraday resonance, we mention here the following results. Friedel *et al.* (1995) showed that in the region $\gamma^2 < \alpha^2 + \beta^2$, $\beta < 0$ for decreasing values of α, transition to temporal chaos takes place. In the latter paper a nonlinear theory predicting a period doubling route to chaos was given. It was also proved that for $\gamma < \alpha$ no standing wave type solutions of (1.1) can exist. Chaotic behaviour of parametrically forced surface waves was observed experimentally and theoretically. Ciliberto and Gollub (1985) studied mode coupling experimentally in a circular cylinder, leading to the onset of chaos. Kambe and Umeki (1990) undertook a comparison of the chaotic mode competition and homoclinic chaos. The bifurcation of stationary and time-dependent solutions is investigated by Umeki (1991a). In that paper, the existence of spatio-temporal chaotic states in a certain range of parameters is illustrated numerically. The motion of resonant modes and numerical calculations of the dynamical equations, showing the existence of chaotic attractors are presented by Umeki (1991b). The other interesting mechanism of the transition to a temporal chaos, when the amplitude of the vertical force is modulated, was described theoretically and experimentally in Chen and Wei (1994), Chen and Wei (1996). We mention here also Miles and Henderson (1990) and Friedel *et al.* (1995) where a current overview of the state of art in the theory of parametrically forced surface waves is given.

In spite of the numerous chaotic phenomena taking place in oscillating tanks, the regular waves processes are also of interest, either theoretical or physical. Il'ichev, (1998) established the existence of the new families of regular standing wave solutions to the equation (1.1) in different regions of parameters. The special features of these solutions may play an important role in the formation of non-steady wave patterns.

The present paper is devoted to the investigation of some possible mechanisms of transition to instability of the localized disturbances in the nonlinear Faraday resonance. This investigation is undertaken in order to observe the connection between the existing families of solitary wave type solutions of (1.1) and the characteristic features of transition to instability of initially localized disturbances.

The paper is organized as follows. In section 2 we give briefly the analytical results concerning the existence of solitary type wave families, bifurcating from the critical curve $\beta^2 + \alpha^2 = \gamma^2$ for both signs of β. The details can be found in a theoretical work (Il'ichev, 1998). In section 3 we describe the numerical method and present the results concerning the transition to instability of initially localized disturbances in various parameter domains. In section 4 we discuss our conclusions.

2. Solitary waves, bifurcating from the critical curve

We look for solutions of (1.1) in the form $u(x) = \{r(x) + is(x)\}\exp(i\psi)$, where ψ is a constant parameter. The equation (1.1) then takes the form

$$r_{xx} = \rho_1 r + \kappa_1 s - 2r(r^2+s^2)$$
$$s_{xx} = \kappa_2 r + \rho_2 s - 2s(r^2+s^2) \tag{2.1}$$

where

$$\rho_1 = -(\beta + \gamma\cos 2\psi), \quad \kappa_1 = \alpha + \gamma\sin 2\psi$$
$$\kappa_2 = -(\alpha - \gamma\sin 2\psi), \quad \rho_2 = -(\beta - \gamma\cos 2\psi)$$

We shall be interested in the solutions of (2.1) of a small amplitude in the vicinity of the critical curve $\beta^2 + \alpha^2 = \gamma^2$ in parameter space. Assume

$$\beta^2 + \alpha^2 = \gamma^2 + \mu\rho_2 \tag{2.2}$$

where the parameter μ is small. The range is determined from the center-manifold theorem, which is used to verify reduction of the basic system (2.3) to the system of the lowest order. This reduction, in its turn, is used to obtain the solutions given in the present section (for details see Il'ichev, 1998). The system of equations (2.1) then may be written in the form of a fourth order dynamical system

$$\begin{aligned} &\dot r = p, \quad \dot s = \lambda \\ &\dot p = \frac{\kappa_1\kappa_2}{\rho_2} r + \kappa_1 s + \mu r - 2r\left(r^2 + s^2\right) \\ &\dot\lambda = \kappa_2 r + \rho_2 s - 2s\left(r^2 + s^2\right) \end{aligned} \tag{2.3}$$

where a dot denotes differentiation with respect to x.

2.1. *Case* $\beta < 0$, $\beta^2 + \alpha^2 > \gamma^2$

For $\beta < 0$, $\beta^2 + \alpha^2 > \gamma^2$ (consequently $\mu\rho_2 > 0$) the system of equations (2.3) has one parametric family (parametrized by ψ) of solitary wave solutions at the lowest order in μ given by Il'ichev, (1998):

$$r = \tilde a_0, \quad s = -\frac{\kappa_2}{\rho_2}\tilde a_0, \quad p = \dot{\tilde a}_0, \quad \lambda = -\frac{\kappa_2}{\rho_2}\dot{\tilde a}_0, \quad \tilde a_0 = \pm\left(\frac{\mu}{\Delta\delta}\right)^{1/2} \operatorname{sech}\left[\left(\frac{\mu}{\Delta}\right)^{1/2} x\right] \tag{2.4}$$

where

$$\delta = \left(1 + \frac{\kappa_2^2}{\rho_2^2}\right), \quad \Delta = \left(1 + \frac{\kappa_1\kappa_2}{\rho_2^2}\right).$$

The solution (2.4) is of order of $|\mu|^{1/2}$. It approximates the solitary wave solution of the full system (2.3) to terms of order $O(\mu)$ (see Theorem 1 in Il'ichev, 1998). For $\kappa_2 = 0$ this solution becomes the known solitary wave solution of (1.1) with fixed $\psi = \psi_2$, given by (1.2) in the introduction. This solitary wave, as was mentioned, is an unstable one. In Il'ichev, (1998) it was conjectured that all this family is unstable. The numerical experiments reported below in section 3 do not contradict this conjecture.

The solitary wave solutions of the lowest order given by (2.4) appear as a result of the generic type of a reversible bifurcation, namely, for the critical value of the parameters ($\mu = 0$) the only imaginary eigenvalue of the linearized right hand side of (2.3) is a double zero. For $\mu\rho_2 > 0$ the eigenvalues are real (which corresponds to solitary waves), for $\mu\rho_2 < 0$ they are imaginary (periodic solutions).

2.2. *Case* $\beta > 0$, $\beta^2 + \alpha^2 < \gamma^2$

In (2.3) make the following substitutions

$$r = a_0 + \kappa_1\left(a_+ + a_-\right), \quad s = -\frac{\kappa_2}{\rho_2} a_0 + \rho_2\left(a_+ + a_-\right), \quad a_1 = \dot a_0 \quad a_+ = \overline{a}_-.$$

The system of equations (2.3) then takes the form, in the new variables,

(2.5)
$$\begin{aligned}
\dot a_0 &= a_1\\
\dot a_1 &= \frac{\mu}{\Delta}\,a_0 - 2\delta a_0^3 + \mu\,\frac{\kappa_1}{\Delta}\,(a_+ + a_-) - 4\,(\kappa_1-\kappa_2)\,a_0^2\,(a_+ + a_-) - 2ca_0\,(a_+ + a_-)^2\\
\dot a_+ &= i\sigma a_+ + i\left\{-\mu\,\frac{\kappa_2}{\Lambda}\,a_0 - \mu\,\frac{\kappa_1\kappa_2}{\Lambda}\,(a_+ + a_-) + \frac{2a_0^2}{\sigma}\,\delta\,(a_+ + a_-)\right.\\
&+ \left.\frac{2}{\sigma}\,(\kappa_1-\kappa_2)\,a_0\,(a_+ + a_-)^2 + \frac{1}{\sigma}\,c\,(a_+ + a_-)^3\right\}\\
\sigma^2 &= -\Delta\rho_2 > 0,\quad \Lambda = -2\sigma^3\rho_2,\quad c = \rho_2^2 + \kappa_1^2.
\end{aligned}$$

Further we consider only the case when μ/Δ is positive, i.e. $\mathrm{sgn}\rho_2 = -\mathrm{sgn}\mu$ in (2.2). From (2.3) it follows that $\beta^2 + \alpha^2 < \gamma^2$, so we are in the domain of instability of the quiescent state.

The advantage of the change of variables consists in the decomposition of the unknowns into the long wave part a_0 and the large wavenumber part a_+. Make the further scaling transformations

$$a_0 = |\mu|^{1/2}\alpha_0\,(\xi),\quad a_1 = |\mu|\alpha_1,\quad a_+ = |\mu|^{3/2}z\,(x),\quad \xi = |\mu|^{1/2}x.$$

The system of equations (2.5) is equivalent to the following (prime denotes differentiation with respect to ξ):

(2.6)
$$\begin{aligned}
\alpha_0' &= \alpha_1\\
\alpha_1' &= \frac{1}{|\Delta|}\,\alpha_0 - 2\delta\alpha_0^3 + 0\,(\mu)\\
\dot z &= i\sigma z - \frac{i\kappa_2}{\Lambda}\,\alpha_0 + 0\,(\mu).
\end{aligned}$$

It follows immediately from (2.6) that at the lowest order in μ the first pair of the equations in (2.5) has the solution (2.4), which also approximates the solution of the full system (2.5) to terms of order $O(\mu)$. In the particular case $\kappa_2 = 0$ the system of equations (2.5) admits the solution $a_+ = 0$ and (2.4) becomes the exact solution of the full system (2.1). This is the known solitary wave solution with $\psi = \psi_1$ and $\rho = (\tilde\mu)^{1/2}$ given by (1.2), where $\tilde\mu = -\beta + \sqrt{\gamma^2 - \alpha^2} > 0$ is small. Yet this solitary wave solution, lying in the region of instability of the quiescent state, can not itself be stable.

The asymptotics of the symmetric travelling wave – the generalized solitary wave (the principal part of it – a solitary wave core – is given by (2.4)) are given by Il'ichev (1998) as

(2.7)
$$\begin{aligned}
r &= \kappa_1 D\sqrt{|\mu|}\exp\left(-\frac{\pi}{2}\,\sigma\sqrt{\frac{\Delta}{\mu}}\right)\sin\sigma x + O\left[\mu\exp\left(-\frac{\pi}{2}\,\sigma\sqrt{\frac{\Delta}{\mu}}\right)\right]\\
s &= \rho_2 D\sqrt{|\mu|}\exp\left(-\frac{\pi}{2}\,\sigma\sqrt{\frac{\Delta}{\mu}}\right)\sin\sigma x + O\left[\mu\exp\left(-\frac{\pi}{2}\,\sigma\sqrt{\frac{\Delta}{\mu}}\right)\right],\quad x\to\infty.
\end{aligned}$$

The generalized solitary wave is a product of nonlinear interaction of a long wave and a high-wavenumber wave. In this case all 4 eigenvalues of the linearized right hand side of (2.3) for $\mu = 0$ are on the imaginary axis, a double zero and two simple complex conjugate eigenvalues. The bifurcation occurs when μ changes sign, then eigenvalues move in pairs to the imaginary axis, or come away from it. The generalized solitary wave corresponds to two real and two imaginary eigenvalues.

3. Numerical simulations

We have investigated the dynamics of solitary waves using the following two-layer explicit numerical scheme

$$i\,\frac{(u_i^{n+1}-u_i^n)}{\Delta t}+\frac{u_{i+1}^n+u_{i-1}^n-2u_i^n}{\Delta x^2}+\beta u_i^n+2|u_i^n|^2u_i^n+i\alpha u_i^n+\gamma u_i^{*n}=0. \tag{3.1}$$

Here the indices n and $n+1$ denote two times steps $t=n\Delta t$ and $t=(n+1)\,\Delta t$, Δx being the spatial step. The scheme (3.1) is a generalization of that for the linear Schrödinger equation described in Samarskii (1977).

The approximation given by the scheme (3.1) is of the first order in Δt and the second order in Δx. The stability condition for (3.1) is found to be $\Delta t < c\Delta x^4$, where c is some constant. We can interpret the numerical solution as a solution of some other equation with higher-order even derivatives in x (Bakholdin, 1994). Additional nondissipative terms can be expressed as follows

$$\sum_{i=2}^{\infty} b_{2i}\,\frac{\partial^{2i}u}{\partial x^{2i}},\quad b_{2i}=O\,(\Delta x^{2i-2}).$$

Scheme (3.1) approximates this equation with $O\,(\Delta x^4)$ order of accuracy (the stability condition was taken into account). That is why this approximation conserves most of the qualitative properties of the initial equation. Effects caused by higher order derivatives are predictable and can be controlled by variation of the parameter Δx.

The numerical method based on (3.1) was verified for the case of evolution of stable solitary waves given by (1.2) with $\psi=\psi_1$. The application of (3.1) in this case gives the standing solitary wave, coinciding with that given by (1.2).

We checked our calculations by variation of the spatial step Δx, and obtained similar results. We also verified the results presented here by use of other numerical schemes. In particular, an explicit predictor-corrector type scheme with second order of accuracy for the t variable has been used:

$$i\,\frac{(u_i^{n+1}-u_i^n)}{\Delta t}+\frac{u_{i+1}^{n+1/2}+u_{i-1}^{n+1/2}-2u_i^n}{\Delta x^2}+\beta u_i^{n+1/2}+2|u_i^{n+1/2}|^2u_i^{n+1/2}+i\alpha u_i^{n+1/2}+\gamma u_i^{*n+1/2}=0$$

$$i\,\frac{(u_i^{n+1/2}-u_i^n)}{\Delta t/2}+\frac{u_{i+1}^n+u_{i-1}^n-2u_i^n}{\Delta x^2}+\beta u_i^n+2|u_i^n|^2u_i^n+i\alpha u_i^n+\gamma u_i^{*n}=0$$

This scheme allows calculations with a smaller value of Δx because the less rigorous condition $\Delta t=c\Delta x^2$ can be used. No essential difference with the results reported here has been found.

Note also that the calculations based on the three-layer scheme (which works in the non-dissipative case) with

$$\frac{\partial u}{\partial t}=\frac{u^{n+1}-u^{n-1}}{2\Delta t},\quad \Delta t < c\Delta x^2$$

as used in Bakholdin (1994) and Bakholding and Il'ichev, (1996) failed in our case because of the increase with time of the divergence between values of two successive steps. However, for short periods of time these two schemes give similar results.

The initial data was taken in such a form that $x=0$ corresponds to the top of a solitary wave crest. In all figures only the region $x>0$ is shown because of the symmetry of the solutions. The qualitative behaviour of the solutions does not depend on the values of α, β and γ provided that the corresponding inequalities hold.

In Figure 1 we show the dependence on t of the amplitude's maximum for initial solitary wave data of moderate amplitude, given by (2.4), in the parameter region $\beta < 0$, $\beta^2 + \alpha^2 > \gamma^2 > \alpha^2$. Curve 1 shows evolution of the amplitude of the solitary wave with $\psi = \psi_2$, $\mu = \tilde{\mu} = -\beta + \sqrt{\gamma^2 - \alpha^2}$ (formula (1.2)). For $\mu > \tilde{\mu}$ (curve 2) an initial solitary wave transforms into the stable solitary wave with $\psi = \psi_1$. For $\mu < \tilde{\mu}$ solitary waves decay. In this region no illuminated waves were detected. Only transformations of solitary wave profiles were observed.

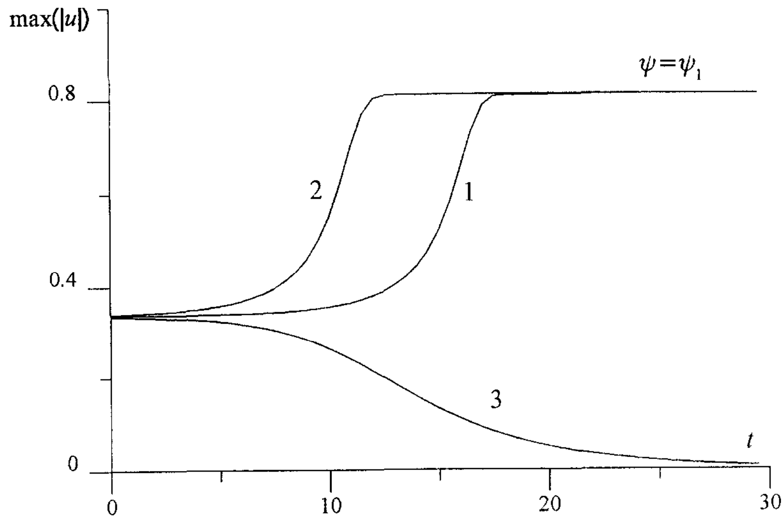

Fig. 1. – Graphs of $\max|u(t)|$ for the case of solitary wave initial data and stable quiescent state ($\beta < 0$, $\beta^2 + \alpha^2 > \gamma^2 > \alpha^2$). Here $\alpha = 1$, $\beta = -0.4$, $\gamma = 1.04$, $\mu = \tilde{\mu} = 0.11344$ (curve 1), $\mu = 0.12344$ (curve 2), $\mu = 0.10344$ (curve 3).

In the region of the unstable quiescent state $\beta < 0$, $\beta^2 + \alpha^2 < \gamma^2$, the decay of the solitary wave (2.4) is shown in Figure 2. In this case the x-domain occupied by the unstable wave increases rapidly.

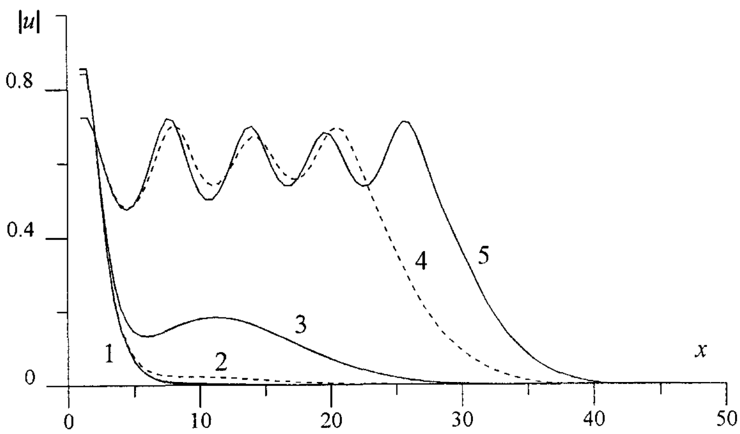

Fig. 2. – Graphs of $|u(t)|$ for the case of solitary wave initial data and the unstable quiescent state ($\beta < 0$, $\beta^2 + \alpha^2 < \gamma^2$). Here $\alpha = 1$, $\beta = -0.3$, $\gamma = 1.1$, $\mu = 0.1$, $T = 5$ (curve 1), $T = 10$ (curve 2), $T = 20$ (curve 3), $T = 30$ (curve 4), $T = 45$ (curve 5).

The development of initial solitary wave disturbances in the parameter region $\beta > 0$, $\gamma^2 > \alpha^2 + \beta^2$ for different moments of time is shown in Figures 3, 4. It is the region where generalized solitary waves exist. These waves in the principal part are given by (2.4) having asymptotics (2.11) at infinity. Il'ichev (1997) conjectured

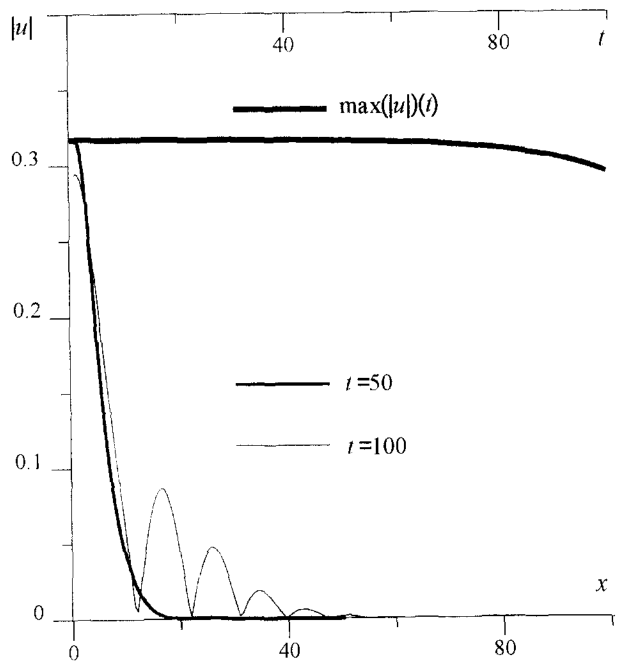

Fig. 3. – First stage of the transition to instability of a solitary wave (2.4), $\alpha = 1$, $\beta = 0.1$, $\gamma = 1.02$, $\mu = 0.1$.

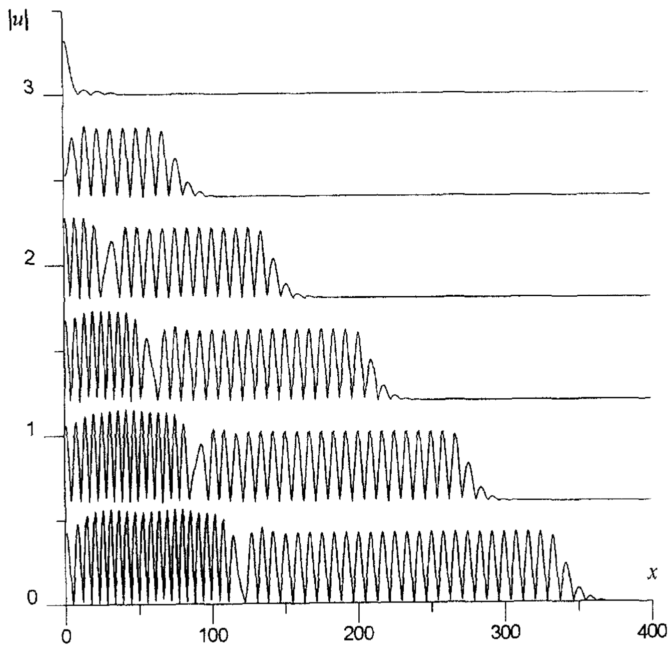

Fig. 4. – Developing instability of solitary wave disturbance, $\alpha = 1$, $\beta = 0.1$, $\gamma = 1.02$, $\mu = 0.1$. The pictures of wave patterns at different instants are placed along the vertical axis: the first picture from above corresponds to $T = 10$, the second $-T = 200$, the third $-T = 300$, the fourth $-T = 400$, the fifth $-T = 500$, the sixth $-T = 600$.

that the existence of generalized solitary waves has a certain influence on the development of the instability of localized disturbances, at least at an initial stage. Figure 3 shows that the first stage of the transition of instability is accompanied by the appearance of periodic waves with a wave number close to σ in (2.7).

The amplitude of the generated wave quickly grows because of instability.

Figure 4 illustrates the further development of instability.

This process leads to the appearance of stable wave states and wave jumps between them. Each of these wave jumps moves with a certain speed. By variation of the spatial size of the mesh it was found that the structure of the first jump, moving with the greatest speed is of a regular character. As in the case shown in Figure 2 the wave amplitude is bounded by some value from above. Wave jumps appear not simultaneously, but step by step during the process of the development of instability.

In conclusion of this section we note that finite difference schemes were used for investigation of the Schrödinger equation in a number of papers (see e.g. Ablowitz and Herbst, 1990, McLaughlin and Schober, 1992, Ablowitz *et al.*, 1996). In these studies the nonlinear Schrödinger equation was solved over a finite interval with periodic boundary conditions. The finite interval and periodic boundary conditions in this case imply the existence of a temporal homoclinic structure in the original equation. The discretization of the original problem leads to an equation which appears to be a perturbation of the original equation. Sometimes, for this perturbed equation the transverse intersection of stable and unstable manifolds can occur, which implies chaos. This chaos is a result of a numerical instability and has an artificial character. The instability disappears when the discretization mesh is refined. In McLaughin and Schober (1992) the effect of numerical stochastization was explained analytically: the Melnikov function was constructed for the perturbed problem and it was found that it has a nondegenerate zero.

4. Conclusion and discussion

In this paper we consider the characteristic features of transition to instability of some localized wave disturbances in a periodically forced long rectangular tank. As a model equation for the surface waves under consideration equation (1.1) as derived by Miles, (1984) is used. This equation is obtained under the assumption of small forcing amplitude, closeness of the forcing frequency to twice the natural frequency, and small wave amplitude in comparison with the width of the tank. Yet, the smallness of the wave amplitude is of a lower order than the smallness of the frequency difference and forcing amplitude, so, in principle, one can consider the waves of a moderate amplitude within the framework of the theory, governed by equation (1.1).

The transition to instability is of a different character in different parameter regions. It was found that evolution of solitary wave type disturbances is influenced by the type of steady wave solutions, existing in a certain parameter region.

In the region $\beta < 0$, $\beta^2 + \alpha^2 > \gamma^2$, where the quiescent state is stable, there exists a family of solitary wave solutions of (1.1), at least close to the critical curve $\alpha^2 + \beta^2 = \gamma^2$ (Il'ichev, 1998). By use of the numerical method described in section 3 we established that the transition to instability of a solitary wave disturbance takes place without radiation. The conjecture made by Il'ichev, (1998) that the solitary wave family (2.4) is unstable, found no contradiction here. Moreover, there exists two different regimes of transition to instability: solitary wave disturbances with amplitudes above a critical value are transformed into the stable solitary wave (1.2). Solitary waves having amplitudes below the critical value decay with time.

In the parameter region $\beta < 0$, $\beta^2 + \alpha^2 < \gamma^2$, where the quiescent state is unstable, the final product of instability is a periodic wave (see Figure 2). We reproduce here the result of Laedke and Spatschek, (1991), concerning the spatial character of the development of instability in this case. And again, the character of the instability is connected, in a certain sense, with steady wave structures existing in this parameter region, namely with the family of cnoidal periodic waves (Miles, 1984; Il'ichev, 1998).

The transition to instability of solitary wave disturbances in the parameter region $\beta > 0$, $\beta^2 + \alpha^2 < \gamma^2$ is also found to be influenced by the features of steady wave patterns taking place in this region. Namely, the existence of the generalized solitary wave family, the principal part given by (2.4) with asymptotics (2.7), causes the radiation of periodic wave tails during the first stage of instability (see Figure 3). In a certain sense, this situation is similar to that taking place in reversible dispersive media: when a system possesses generalized solitary waves, no localized disturbance can remain steady, or generate waves, they decay by radiation of periodic wave tails. The transition to instability reported here has such a character only during the first stage. The result of the developed instability of a solitary wave disturbance, shown in Figure 4, presents the stable wave states, connected by a number of jumps with structure.

Acknowledgements. – The research of the first author is supported by the INTAS Grant No. 95-0435 and by Foundation for Fundamental Research Grant (Project No. 96-0100991). The research of the second author is supported by a Research Fellowship from the Alexander von Humboldt Foundation.

REFERENCES

ABLOVITZ M. J., HERBST B. M., 1990, On homoclinic structure and numerically induced chaos for the nonlinear Schrödinger equation, *SIAM J. Appl. Math.*, **50**, 339–351.

ABLOVITZ M. J., HERBST B. M., SCHOBER C. M., 1996, Computational chaos in the nonlinear Schrödinger equation without homoclinic crossings, *Physica A*, **228**, 212–235.

BAKHOLDIN I. B., 1994, Wave jumps in media described by a modified Schrödinger equation, *Fluid Dynamics*, **29**, 528–538.

BAKHOLDIN I., IL'ICHEV A., 1996, Radiation and modulational instability described by the fifth-order Korteweg-de Vries equation, Contemporary Mathematics, **200**, 1–15.

BARASHENKOV I. V., BOGDAN M. M., KOROBOV V. I., 1991, Stability diagram of the phase-locked solitons in the parametrically driven, damped nonlinear Schrödinger equation, *Europhys. Lett.*, **15**, 113–118.

BENILOV E. S., GRIMSHAW R., KUZNETSOVA E. P., 1993, The generation of radiating waves in a singularly-perturbed Kortewed-de Vries equation, *Physica D*, **69**, 270–278.

CHEN X.-N., WEI R., 1994, Dynamic behaviour of a non-propagating soliton under a periodically modulated oscillation, *J. Fluid Mech.*, **259**, 291–303.

CHEN X.-N., WEI R., 1996, Chaotic motion of a non-propagating soliton through a period-doubling route, in *Nonlinear acoustics in perspective*, ed. R. J. Wei, 353–361, Nanjing University Press.

CILIBERTO S., Gollub J. P., 1985, Chaotic mode competition in parametrically forced surface waves, *J. Fluid. Mech.*, **158**, 381–398.

FRIEDEL H., LAEDKE E. W., SPATSCHEK K. H., 1995, Bifurcations and nonlinear dynmics of surface waves in Faraday resonance, *J. Fluid Mech.*, **284**, 341–358.

IL'ICHEV A., 1998, Faraday resonance: asymptotic theory of surface waves, to appear in *Physica D*.

KAMBE T., UMEKI M., 1990, Nonlinear dynamics of two-mode interactions in parametric excitation of surface waves, *J. Fluid Mech.*, **212**, 373–393.

LAEDKE E. W., SPATSCHEK K. H., 1991, On localized solutions in nonlinear Faraday resonance, *J. Fluid Mech.*, **223**, 589–601.

LAZZARA A., PUTTERMAN S., 1984, Theory of non-propagating surface-wave solitons, *J. Fluid Mech.*, **148**, 443–449.

McLAUGHLIN D. W., SCHOBER C. M., 1992, Chaotic and homoclinic behaviour for numerical discretizations of the nonlinear Schrödinger equation, *Physica D*, **57**, 447–465.

MILES J., 1984, Parametrically exited solitary waves, *J. Fluid Mech.*, **148**, 451–460.

MILES J., HENDERSON D., 1990, Parametrically forced surface waves, *Ann. Rev. Fluid Mech.*, **22**, 143–165.

SAMARSKII A. A., 1977, *Theory of difference schemes*, Nauka, Moskow, USSR.

UMEKI M., 1991a, Parametric dissipative nonlinear Schrödinger equation, *J. Phys. Soc. Jpn.*, **60**, 146–167.

UMEKI M., 1991b, Faraday resonance in rectangular geometry, *J. Fluid Mech.*, **227**, 161–192.

WU J., KEOLIAN R., RUDNIK I., 1984, Observation of non-propagating hydrodynamic soliton, *Phys. Rev. Lett.*, **52**, 1421–1424.

(Received 16 June 1997;
revised 30 April 1998;
accepted 13 May 1998.)